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FORMULÆ FOR INTEGRATION.

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The following formulæ are readily derived from those of the text-books, and have the advantage of being only two instead of four in number, of being so symmetrical as to be easily remembered, and of being easy of application; and also of performing by a single operation what has hitherto required successive applications of formulæ.

To integrate

$$\int x^m (a + bx^n)^p \, dx.$$

We may term this expression the *Integrand.*

Substitute φ for the part within parenthesis. Thus the integrand becomes

$$\int x^m \varphi^p dx, \quad \text{where} \quad \varphi = a + bx^n.$$

I. To lower m, positive or negative.

This includes formulæ A and B of the text-books.

$$\int [b(m + np + 1)x^m + a(m - n + 1)x^{m-n}] \varphi^p dx = x^{m-n+1}\varphi^{p+1}. \qquad \text{(A)}$$

From this we derive a series of equations by substituting $m - n$ for m at each stage when m is positive, and by substituting $m + n$ for m negative, so as numerically to lower the index of x, and usually to make it integrable. The general expression for the $(k + 1)$th of these equations is

$$\int [b(m + np + 1 \mp kn) x^{m \mp kn} + a(m - n + 1 \mp kn) x^{m-n \mp kn}] \varphi^p dx = x^{m-n+1 \mp kn} \varphi^{p+1}.$$

(Observe that the two quantities on the right side are the last two quantities of the left side with the indices increased by unity.)

II. To lower p, positive or negative.

This includes formulæ C and D of the text-books.

$$\int [(m + np + 1)\varphi^p - anp\varphi^{p-1}] x^m dx = x^{m+1}\varphi^p. \qquad \text{(C)}$$

From this we derive a series of equations by substituting $p - 1$ for p positive, and $p + 1$ for p negative. The general expression for the $(k + 1)$th equation is

$$\int [(m + np + 1 \mp k)\varphi^{p \mp k} - an(p \mp k)\varphi^{p \mp k - 1}] x^m dx = x^{m+1}\varphi^{p \mp k}.$$

III. To perform successive integrations by a single operation.

We submit a set of the derived equations, the upper ones (A_{-2}), (A_{-1}) for m negative, the lower ones (A_1), (A_2) for m positive. By comparing them we can find the law of their constitution:

$$\int [b(m+2n+np+1)x^{m+2n} + a(m+n+1)x^{m+n}]\,\varphi^p dx = x^{m+n+1}\varphi^{p+1}; \quad (A_{-2})$$

$$\int [b(m+n+np+1)x^{m+n} + a(m+1)x^{m}]\,\varphi^p dx = x^{m+1}\varphi^{p+1}; \quad (A_{-1})$$

$$\int [b(m+np+1)x^{m} + a(m-n+1)x^{m-n}]\,\varphi^p dx = x^{m-n+1}\varphi^{p+1}; \quad (A)$$

$$\int [b(m-n+np+1)x^{m-n} + a(m-2n+1)x^{m-2n}]\,\varphi^p dx = x^{m-2n+1}\varphi^{p+1}; \quad (A_1)$$

$$\int [b(m-2n+np+1)x^{m-2n} + a(m-3n+1)x^{m-3n}]\,\varphi^p dx = x^{m-3n+1}\varphi^{p+1}. \quad (A_2)$$

In Equations (A), (A_1), (A_2), etc., assume r_1, r_2, etc., to represent the ratios of the coefficients of the same powers of x in the successive equations; thus:

$$r_1 = \frac{a(m-n+1)}{b(m-n+np+1)}$$

$$r_i = \frac{a}{b}\,\frac{(m-in+1)}{(m-in+np+1)}.$$

Then, from equations (A), (A_1), and (A_2), successively,

$$\begin{aligned}
\int b(m+np+1)x^m\varphi^p dx &= \varphi^{p+1}x^{m-n+1} - \int a(m-n+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}x^{m-n+1} \\
&\quad - \int \frac{a(m-n+1)}{b(m-n+np+1)}\cdot b(m-n+np+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}x^{m-n+1} - r_1\int b(m-n+np+1)x^{m-n}\varphi^p dx \\
&= \varphi^{p+1}(x^{m-n+1} - r_1x^{m-2n+1}) \\
&\quad + r_1\int a(m-2n+1)x^{m-2n}\varphi^p dx \\
&= \varphi^{p+1}(x^{m-n+1} - r_1x^{m-2n+1} + r_1r_2x^{m-3n+1}) \\
&\quad + r_1r_2\int a(m-3n+1)x^{m-3n}\varphi^p dx.
\end{aligned}$$

Hence,

$$\int x^m\varphi^p dx = \frac{\varphi^{p+1}}{b(m+np+1)}\left[\begin{matrix} x^{m-n+1} - r_1x^{m-2n+1} + r_1r_2x^{m-3n+1} \\ + \ldots + (-1)^{l-1}r_1r_2 - r_{l-1}x^{m-ln+1} \end{matrix}\right],$$

where l is the integral part of the quotient $\frac{m+1}{n}$.

Example 1. Integrate $\int x^{11}(a+bx^2)^{-\frac{1}{2}}dx$.

$$\int x^{11}(a+bx^2)^{-\frac{1}{2}}dx = \frac{\sqrt{a+bx^2}}{11b}\left[x^{10} - \frac{10}{9}\frac{a}{b}x^8 + \frac{10.8}{9.7}\frac{a^2}{b^2}x^6 - \frac{10.8.6}{9.7.5}\frac{a^3}{b^3}x^4 + \frac{10.8.6.4}{9.7.5.3}\frac{a^4}{b^4}x^2 - \frac{10.8.6.4.2}{9.7.5.3.1}\frac{a^5}{b^5}\right].$$

The integration is, in this case, completed, as the next term would have zero in the numerator. In the following example zero would appear in the denominator, proving that the complete integration cannot be reached by this method. But we may remove the integrand-remainder outside the brackets, and solve it by a different method.

Example 2. $\int x^{10}(a+bx^2)^{-\frac{1}{2}}dx$

$$= \frac{\sqrt{a+bx^2}}{10b}\left[x^9 - \frac{9}{8}\frac{a}{b}x^7 + \frac{9.7}{8.6}\frac{a^2}{b^2}x^5 - \frac{9.7.5}{8.6.4}\frac{a^3}{b^3}x^3 + \frac{9.7.5.3}{8.6.4.2}\frac{a^4}{b^4}x\right] - \frac{9.7.5.3.1}{10.8.6.4.2}\frac{a^5}{b^5}\int(a+bx^2)^{-\frac{1}{2}}dx.$$

This last integrand is otherwise found to be

$$\frac{1}{\sqrt{b}}\log\left[x + \sqrt{\frac{a}{b} + x^2}\right].$$

IV. For m negative we proceed from equation (A_1), reversing our course, and therefore r_1, r_2, etc. will be in the reciprocal order.

Example 3. $$\int x^{-10}(a+bx^2)^{-\frac{1}{2}}dx = -\frac{\sqrt{a+bx^2}}{9a}\left[x^{-9} - \frac{8}{7}\frac{b}{a}x^{-7} + \frac{8.6}{7.5}\frac{b^2}{a^2}x^{-5} - \frac{8.6.4}{7.5.3}\frac{b^3}{a^3}x^{-3} + \frac{8.6.4.2}{7.5.3.1}\frac{b^4}{a^4}x^{-1}\right].$$

V. To lower p positive, by a single operation, the formula is

$$\int \varphi^p x^m dx = \frac{x^{m+1}}{m+np+1}(\varphi^p + r_1\varphi^{p-1} + r_1r_2\varphi^{p-2} + \ldots),$$

where

$$r_i = \frac{an(p-i+1)}{m-in+np+1}.$$

VI. To lower p negative, by a single operation, we obtain

$$\int \varphi^p x^m dx = -\frac{x^{m+1}}{an(p+1)}(\varphi^{p+1} + r_1\varphi^{p+2} + r_1r_2\varphi^{p+3} + \ldots),$$

where

$$r_i = \frac{m+in+np+1}{an(p+i+1)}.$$